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Author(s)	Kim, Tae Hwa
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APPROXIMATION OF COMMON FIXED POINTS FOR A FAMILY OF NON-LIPSCHITZIAN SELF-MAPPINGS

TAE HWA KIM

ABSTRACT. In the present paper, we first give some examples of self-mappings which are of strongly asymptotically nonexpansive type, not strictly hemicontractive, but satisfy the property (H). It is then shown that the modified Mann and Ishikawa iteration processes for a family $\Im = \{T_n : n \in \mathbb{N}\}$ of self-mappings $T_n : K \to K$, defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n$ and $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n[(1-\beta_n)x_n + \beta_n T_n x_n]$, respectively, converge strongly to the unique common fixed point of such a family $\Im$ in general Banach spaces.

1. PRELIMINARIES

Let X be a real Banach space and X^* the dual space of X. Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. The norm of X is said to be *Gâteaux differentiable* (and X is said to be *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in U. It is said to be *uniformly Gâteaux differentialbe* if for each $y \in U$, this limit is attained uniformly for $x \in U$. The norm is said to be *Fréchet differentiable* if for each $x \in U$, the limit is obtained uniformly for $y \in U$. Finally, the space X is said to have a *uniformly Fréchet differentiable* norm (and X is said to be *uniformly smooth*) if the limit is attained uniformly for $(x, y) \in U \times U$.

The normalized duality mapping J from X into the family of nonempty subset of X^* is defined by

$$J(x) = \{f \in X^* : \|f\|^2 = \|x\|^2 = \langle x, f\rangle\},$$

where $\langle x, f\rangle$ denotes the value of f at x. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Moreover, it is known that J is single valued if and only if X is smooth, while if X is uniformly smooth, then the mapping J is uniformly continuous on bounded sets.

Let X be a real Banach space and let K be a nonempty subset of X (not necessarily convex) and $T : K \to K$ a self mapping of K. There appear in the literature two definitions of an asymptotically nonexpansive mapping. The weaker definition (cf. Kirk[19]) requires that

$$\limsup_{n\to\infty} \sup_{y\in K} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

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for every $x \in K$ and that T^N is continuous for some $N \geq 1$. The stronger definition (briefly called *asymptotically nonexpansive* as in [15]) requires each iterate T^n to be Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. For further generalization of an averaging iteration of Schu [25], Bruck et al. [4] introduced a definition somewhere between these two : T is *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x-y\|) \leq 0. \tag{1.1}$$

In this paper, we consider the self mapping of K satisfying only (1.1) without the assumption of uniform continuity of T. Throughout we shall refer to such a mapping as *strongly asymptotically nonexpansive type.*

A mapping $T : K \to X$ is said to be *pseudo-contractive* [26] if for all $x, y \in K$ there exists $j \in J(x-y)$ such that

$$\langle Tx - Ty, j\rangle \leq \|x-y\|^2.$$

In [18], Kato discovered the relationship between pseudocontractive mappings and accretive mappings, proving

Lemma 1.1 [18]. *Let $x, y \in X$. Then $\|x\| \leq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there exists $j \in J(x)$ such that $\langle y, j\rangle \geq 0$.*

Applying Lemma 1.1, we know that a mapping T is pseudocontractive if and only if $(I-T)$ is accretive, i.e., the inequality

$$\|x-y\| \leq \|x - y + r\{(I-T)x - (I-T)y\}\|$$

holds for all $x, y \in K$ and all $r \geq 0$.

In the sequel, we need the following two lemmas for the proof of our main results. The first is actually Lemma 1 of Petryshyn [23] and the second is Lemma 2 of Liu [21]. For the first result, Asplund [1] also proved a general result for single-valued duality mappings, which can be used to derive this lemma and more recently this lemma was revisited by Haiyun-Yuting [16].

Lemma 1.2 [23]. *For any $x, y \in X$ and $j \in J(x+y)$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j\rangle.$$

Lemma 1.3 [21]. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n$$

with $\{t_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} a_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

A mapping $T : K \to X$ is said to be *strictly pseudo-contractive* [8], [26] (or *strong pseudo-contraction* [9]) if there exists $t > 1$ such that for all $x, y \in K$ there exists $j \in J(x-y)$ such that

$$\mathrm{Re}\langle Tx - Ty, j\rangle \leq \frac{1}{t}\|x-y\|^2.$$

Let $F(T)$ denotes the set of all fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$. If $F(T) \neq \emptyset$, the mapping $T : K \to X$ is said to be *strictly hemicontractive* [8] if there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)$ there exists $j \in J(x - x^*)$ such that

$$\langle Tx - x^*, j\rangle \leq \frac{1}{t}\|x - x^*\|^2. \tag{1.2}$$

Using Lemma 1.1, it is easy to check [8] that the strict hemicontractivity of T is equivalent to the following inequality

$$\|x - x^*\| \leq \|(1+r)(x - x^*) - rt(Tx - x^*)\|$$

holds for all $x \in K$, $x^* \in F(T)$ and $r > 0$.

For an example of a Lipschitzian self-mapping which is not strictly pseudocontractive but strictly hemicontractive, see [8].

Motivated by the definition of strict hemicontractivity, we can consider a mapping $T : K \to K$ satisfying the following property, i.e., there exists $t > 1$ such that for all $x \in K$ and $x^* \in F(T)(\neq \emptyset)$ there exists $j \in J(x - x^*)$ such that

$$\text{(H)} \qquad \limsup_{n\to\infty} \langle T^n x - x^*, j\rangle \leq \frac{1}{t}\|x - x^*\|^2.$$

Note that any mapping $T : K \to K$ which is both strictly hemicontractive and asymptotically nonexpansive satisfies the property (H). Indeed, since T is strictly hemicontractive and asymptotically nonexpansive, we have

$$\langle T^n x - x^*, j\rangle \leq \frac{1}{t}\|T^{n-1}x - x^*\|^2 \leq \frac{1}{t}L_n^2\|x - x^*\|^2.$$

Taking lim sup on both sides, since $L_n \to 1$ as $n \to \infty$, T satisfies (H).

First we give two examples of the discontinuous self-mappings which are strongly asymptotically nonexpansive type, not strictly hemicontractive, but satisfies the above property (H).

Example 1.1. Let $X = \mathbb{R}$ with the usual norm $|\cdot|$ and let $K = [0,1]$. Let $a_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then, construct a discontinuous mapping T as follows. On the each subinterval $[a_{n+1}, a_n]$, the graph of T consists of all rational numbers of the sides of the isosceles triangle with base $[a_{n+1}, a_n]$ and height a_{n+1} and zeros for irrational numbers in K. Thus, $Ta_n = 0$ and, if x_n denotes the midpoint of $[a_{n+1}, a_n]$, then $Tx_n = a_{n+1}$. If we further define $T0 = 0$, $T : K \to K$ is not continuous but clearly $F(T) = \{0\}$. Since $T^n x \to 0$ uniformly as $n \to \infty$, T is strongly asymptotically nonexpansive type. Obviuosly, T satisfies the property (H) but is not strictly hemicontractive.

Example 1.2. Let $K = [0,1] \subseteq \mathbb{R}$ and define $Tx = \frac{1}{4}$ if $x = \frac{1}{4}, 1$, $Tx = 1$ for $x \in [0, \frac{1}{2}] \setminus \frac{1}{4}$, and $Tx = \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Note that for all $x \in K$, $T^n x = \frac{1}{4} \in F(T) = \{\frac{1}{4}\}$ for $n \geq 3$. Then $T : K \to K$ is a discontinuous mapping of strongly asymptotically nonexpansive type which is not nonexpansive. Obviuosly, T satisfies the property (H). However, T is not *strictly* hemicontractive.

Here we give an example of the discontinuous self-mapping with the property (H) which is strongly asymptotically nonexpansive type, not asymptotically nonexpansive.

Example 1.3. Let $K = [0,1] \subseteq \mathbb{R}$ and let φ be the Cantor ternary function. Define $T : K \to C$ by

$$T(x) = \begin{cases} x/2 & \text{if } 0 \leq x \leq 1/2, \\ \varphi((1-x)/2) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Note that $T^n x \to 0$ uniformly on K. Therefore, T is a discontinuous mapping of strongly asymptotically nonexpansive type with the property (H). But it is not asymptotically nonexpansive because φ is not Lipschizian continuous on $[0, \frac{1}{2}]$. Note that T is also *strictly* hemicontractive.

Recall that a mapping $T : K \to X$ is said to be *strongly accretive* [3] (or [29]) if there exists a positive number k such that for each $x, y \in K$ there is $j \in J(x-y)$ such that

$$\langle Tx - Ty, j\rangle \geq k\|x-y\|^2.$$

Using Lemma K again this is equivalent to

$$\|x-y\| \leq \|x-y+r\{(T-kI)x-(T-kI)y\}\|,$$

for all $r > 0$, where I denotes the identity mapping of X. Without loss of generality, we can assume $k \in (0,1)$. Then it was known [2] that the similar connection between strict pseudocontractivity and strong accretivity is that a mapping $T : K \to K$ is strictly pseudocontractive if and only if $I - T$ is strongly accretive, i.e., the inequality

$$\|x-y\| \leq \|x-y+r\{(I-T-kI)x-(I-T-kI)y\}\| \tag{1.3}$$

holds for any $x, y \in K$ and $r > 0$, where $k = \frac{(t-1)}{t} \in (0,1)$.

It is well known that if $T : K \to X$ is continuous and strictly pseudocontractive, then T has a unique fixed point (see Corollary 1 of Deimling [12]). Furthermore, if $T : X \to X$ is continuous and strongly accretive, then T is surjective, i.e., for a given $f \in X$, the equation $Tx = f$ has a unique solution.

Recently, the convergence problems of Ishikawa and Mann iteration sequences (cf. Ishikawa [17] and Mann [22]) have been studied extensively by many authors (see Chidume [5-8], Chidume and Osilike [9-11], Deng [13], Deng-Ding [14], Haiyun-Yuting [16], Liu [20], Liu [21], Reich [24] and Tan-Xu [27]) for strictly pseudocontractive (or strongly accretive) mappings.

Especially, Liu [20] proved, using the inequality (1.3), that the Mann iteration process converges strongly to the unique fixed point of a Lipschitzian and strictly pseudo-contractive mapping, which extends corresponding results of [5-8], [27] and [29] to the general Banach spaces as follows.

Theorem 1.1 [20]. *Let K be a nonempty closed, convex and bounded subset of a Banach space X and let $T : K \to K$ be Lipschitzian and strictly pseudocontractive mapping. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad x_1 \in K,$$

with $\{\alpha_n\} \subset (0,1]$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0,$$

converges strongly to $q \in F(T)$ and $F(T)$ is a singleton set.

Subsequently, Haiyun-Yuting [16] proved by using Lemma 1.2 that the Ishikawa iteration process converges strongly to the unique fixed point of a continuous and strictly pseudocontrative map without Lipschitz assumption in a real *uniformly smooth* Banach space.

Theorem 1.2 [16]. *Let K be a nonempty closed, convex and bounded subset of a real uniformly smooth Banach space X. Assume that $T : K \to K$ is a continuous strictly pseudocontractive mapping. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences satisfying*

(i) $0 < \alpha_n, \beta_n < 1$ *and* $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ *generated from an arbitrary* $x_1 \in K$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases}$$

converges strongly to the unique fixed point of T.

On the other hand, Chidume and Osilke [9] proved with the similar method of the proof as in [20] that the Ishikawa iteration process also converges strongly to the unique fixed point of a uniformly continuous and stricltly pseudo-contractive mapping in a real Banach space.

Theorem 1.3 [9]. *Let* K *be a nonempty closed, convex and bounded subset of a real Banach space* X. *Suppose* $T : K \to K$ *is a unifomly continuous and strictly pseudocontractive mapping. Then, the sequence* $\{x_n\}_{n=1}^{\infty}$ *generated from an arbitrary* $x_1 \in K$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \geq 1, \end{cases}$$

converges strongly to $q \in F(T)$ *and* $F(T)$ *is a singleton set. Here,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1]$ *satisfying*

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n.$$

In 1995, Liu [21] introduced the Ishikawa iteration process with errors as follows:

$$\text{(1.4)} \qquad \begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \qquad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ such that (i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$, (ii) $\{\beta_n\}$ is bounded, (iii) $\{u_n\}$ and $\{v_n\}$ are summable sequences in X, and T is a Lipschitzian strongly accretive mapping in a uniformly smooth Banach space X.

1n 1998, Xu [28] intoduced the Ishikawa iteration processes emphasizing the randomness of errors as follows:

$$\text{(1.5)} \qquad \begin{cases} x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n = \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}, \{\hat{\gamma}_n\}$ are sequences in $[0,1]$ such that (i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = 0$, (ii) $\lim_{n\to\infty} \hat{\beta}_n = \infty$, (iii) $\lim_{n\to\infty} \hat{\gamma}_n = 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$, (iv) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$, and $\{v_n\}, \{u_n\}$ are bounded suquences in Banach space X, an T is a strongly pseudocontractive mapping in unifor mly smooth Banach space X.

In these respects, it seems natural to ask whether the above theorems are still valid for a family $\Im = \{T_n : n \in \mathbb{N}\}$ of self-mappings $T_n : K \to K$ which satisfies the property (H) type (as the definition replaced T^n in (H) by T_n). For our affirmative argument, consider the similar iteration process with errors of (1.5) as follows:

$$\text{(1.6)} \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T_n y_n + \gamma_n u_n, \\ y_n = \alpha'_n x_n + \beta'_n T_n x_n + \gamma'_n v_n, \qquad n \geq 1, \end{cases}$$

where $\{u_n\}$ and $\{v_n\}$ are two bounded sequence in K; $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1]$ satisfying the conditions

$$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1,$$

for all $n \geq 1$.

Lemma 1.4. *Let K be a nonempty closed and convex subset of a Banach space X. Let two iterative sequences $\{x_n\}$ and $\{y_n\}$ be given as in (1.6) for a family $\Im = \{T_n : n \in \mathbb{N}\}$ of self-mappings $T_n : K \to K$, $n \in \mathbb{N}$. Put $B := \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ $(\subset K)$, $q \in F(\Im) := \cap_{n\in\mathbb{N}} F(T_n)$ and*

$$c_n := \max\{0, \sup_{x\in B}(\|T_n x - q\| - \|x - q\|)\}.$$

Then

$$\|x_n - q\| \le d + 2\sum_{k=1}^{n-1} c_k, \quad \|y_n - q\| \le d + 2\sum_{k=1}^{n-1} c_k + c_n, \tag{1.7}$$

for $n \in \mathbb{N}$, where

$$d := \max\{\sup_{n\ge1}\|u_n - q\|, \sup_{n\ge1}\|v_n - q\|, \|x_1 - q\|\}.$$

Proof. The proof employs mathematical induction. Since $\|x_1 - q\| \le d$ and

$$\begin{aligned}
\|y_1 - q\| &= \|\alpha_1' x_1 + \beta_1' T x_1 + \gamma_1' v_1 - q\| \\
&\le \alpha_1'\|x_1 - q\| + \beta_1'\|Tx_1 - q\| + \gamma_1'\|v_1 - q\| \\
&\le \alpha_1'\|x_1 - q\| + \beta_1'(c_1 + \|x_1 - q\|) + \gamma_1'\|v_1 - q\| \\
&\le (\alpha_1' + \beta_1' + \gamma_1')d + \beta_1' c_1 \\
&\le d + c_1,
\end{aligned}$$

(1.7) holds for $n = 1$. Suppose (1.7) holds for $n = k$, i.e.,

$$\|x_k - q\| \le d + 2\sum_{j=1}^{k-1} c_j, \quad \|y_k - q\| \le d + 2\sum_{j=1}^{k-1} c_j + c_j.$$

Then, for $n = k + 1$, we have

$$\begin{aligned}
\|x_{k+1} - q\| &= \|\alpha_k x_k + \beta_k T_k y_k + \gamma_k u_k - q\| \\
&\le \alpha_k\|x_k - q\| + \beta_k\|T_k y_k - q\| + \gamma_k\|u_k - q\| \\
&\le \alpha_k\|x_k - q\| + \beta_k(c_k + \|y_k - q\|) + \gamma_k\|u_k - q\| \\
&\le \alpha_k(d + 2\sum_{j=1}^{k-1} c_j) + \beta_k c_k + \beta_k(d + 2\sum_{j=1}^{k-1} c_j + c_k) + \gamma_k d \\
&= d + 2(\alpha_k + \beta_k)\sum_{j=1}^{k-1} c_j + 2\beta_k c_k \\
&\le d + 2\sum_{j=1}^{k} c_j
\end{aligned}$$

and

$$\begin{aligned}\|y_{k+1}-q\| &= \|\alpha'_{k+1}x_{k+1}+\beta'_{k+1}T_{k+1}x_{k+1}+\gamma'_{k+1}v_{k+1}-q\| \\ &\le \alpha'_{k+1}\|x_{k+1}-q\|+\beta'_{k+1}\|T_{k+1}x_{k+1}-q\|+\gamma'_{k+1}\|v_{k+1}-q\| \\ &\le \alpha'_{k+1}\|x_{k+1}-q\|+\beta'_{k+1}(c_{k+1}+\|x_{k+1}-q\|)+\gamma'_{k+1}\|v_{k+1}-q\| \\ &\le (\alpha'_{k+1}+\beta'_{k+1})\|x_{k+1}-q\|+\beta'_{k+1}c_{k+1}+\gamma'_{k+1}d \\ &\le (\alpha'_{k+1}+\beta'_{k+1})(d+2\sum_{j=1}^{k}c_j)+\beta'_{k+1}c_{k+1}+\gamma'_{k+1}d \\ &\le d+2\sum_{j=1}^{k}c_j+c_{k+1}.\end{aligned}$$

Therefore, by mathematical induction, (1.7) holds for all $n \in \mathbb{N}$.

2. Main results

We first begin with an easy observation of the property (H) type. The first equivalent is

$$\text{(H}_1\text{)} \qquad \liminf_{n\to\infty}\langle x-T_nx,j\rangle \ge \frac{(t-1)}{t}\|x-x^*\|^2.$$

Let $x \neq x^*$. For a fixed ϵ with $0<\epsilon<\frac{(t-1)}{t}$, it follows from the property (H$_1$) that there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\text{(H}_2\text{)} \qquad \begin{aligned}\langle x-T_nx,j\rangle &\ge (\frac{t-1}{t}-\epsilon)\|x-x^*\|^2 \\ &= k_\epsilon\|x-x^*\|^2,\end{aligned}$$

where $k_\epsilon := (\frac{t-1}{t}-\epsilon) \in (0,1)$. This inequality is obviously equivalent to

$$\text{(H}_3\text{)} \qquad \langle T_nx-x^*,j\rangle \le (1-k_\epsilon)\|x-x^*\|^2, \qquad \forall n \ge n_0.$$

For employing the method of the proof in [20], we need the following equivalent form of the property (H$_2$) by virtue of Lemma 1.1:

$$\text{(H}_4\text{)} \qquad \|x-x^*\| \le \|x-x^*+r\{(I-T_n-k_\epsilon I)x-(I-T_n-k_\epsilon I)x^*\}\|$$

for all $n \ge n_0$ and all $r>0$.

Using the property (H$_3$), Lemma 1.3 and 1.4, we are now ready to present the following

Theorem 2.1. *Let K be a nonempty closed and convex subset of a Banach space X. Suppose a family $\Im=\{T_n : n\in\mathbb{N}\}$ of self-mappings $T_n : K\to K$, $n\in\mathbb{N}$ satisfies the property* (H) *type. Suppose $F(T)\neq\emptyset$ and put*

$$c_n=\max\{0, \sup_{x,y\in K}(\|T_nx-T_ny\|-\|x-y\|)\},$$

so that $\sum_{n=1}^{\infty}c_n<\infty$. Then the modified Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.6) converges strongly to the unique common fixed point of $\Im$ in K, where

$$\text{(i)} \qquad \lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\beta'_n=\lim_{n\to\infty}\gamma'_n=0;$$

(ii) $$\sum_{n=1}^{\infty}\beta_n=\infty \quad and \quad \sum_{n=1}^{\infty}\gamma_n<\infty.$$

Proof. Since $F(T)\neq\emptyset$, take $q\in F(T)$. Lemma 1.4 immediately gives

$$\|x_{n+1}-q\|\le M,\ \|y_{n+1}-q\|\le M,$$

for all $n\in\mathbb{N}$, where $M:=d+2\sum_{n=1}^{\infty}c_n<\infty$. Lemma 1.2 and the property (H$_3$) yields

$$\begin{aligned}\|x_{n+1}-q\|^2&=\|\alpha_n(x_n-q)+\beta_n(T_ny_n-q)+\gamma_n(u_n-q)\|^2\\ &\le\alpha_n^2\|x_n-q\|^2+2\beta_n\langle T_ny_n-q,j_n\rangle+2\gamma_n\langle u_n-q,j_n\rangle\\ &\le\alpha_n^2\|x_n-q\|^2+2\beta_n\langle T_nx_{n+1}-q,j_n\rangle\\ &\quad+2\beta_n\langle T_ny_n-T_nx_{n+1},j_n\rangle+2\gamma_n\langle u_n-q,j_n\rangle\\ &\le\alpha_n^2\|x_n-q\|^2+2\beta_n(1-k_\epsilon)\|x_{n+1}-q\|^2+2\beta_nd_n+2\gamma_nM^2,\end{aligned}\tag{2.1}$$

for $j_n\in J(x_{n+1}-q)$ and for all $n\ge n_0$, where $d_n:=\langle T_ny_n-T_nx_{n+1}\rangle$. We first claim that $j_n\to 0$ as $n\to\infty$. In fact, by the parameter conditions (i) and (ii) we get

$$\begin{aligned}&\|y_n-x_{n+1}\|=\|(y_n-q)+(q-x_{n+1})\|\\ =&\|\alpha_n'(x_n-q)+\beta_n'(T_nx_n-q)+\gamma_n'(v_n-q)\\ &\quad-\alpha_n(x_n-q)-\beta_n(T_ny_n-q)-\gamma_n(u_n-q)\|\\ \le&(|\beta_n'-\beta_n|+|\gamma_n'-\gamma_n|)\|x_n-q\|+\beta_n'\|T_nx_n-q\|\\ &\quad+\gamma_n'\|v_n-q\|+\beta_n\|T_ny_n-q\|+\gamma_n\|u_n-q\|\\ \le&(\beta_n'+\beta_n+\gamma_n'+\gamma_n)\|x_n-q\|+\beta_n'(c_n+\|x_n-q\|)+\gamma_n'\|v_n-q\|\\ &\quad+\beta_n(c_n+\|y_n-q\|)+\gamma_n\|u_n-q\|\\ \le&2(\beta_n'+\beta_n+\gamma_n'+\gamma_n)M+c_n(\beta_{n'}+\beta_n)\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Therefore, since $c_n\to 0$ as $n\to\infty$, we get

$$\begin{aligned}\|T_ny_n-T_nx_{n+1}\|&\le[\|T_ny_n-T_nx_{n+1}\|-\|y_n-x_{n+1}\|]+\|y_n-x_{n+1}\|\\ &\le c_n+\|y_n-x_{n+1}\|\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Since $\|j_n\|=\|x_{n+1}-q\|\le M$, this gives

$$\begin{aligned}|d_n|&=|\langle T_ny_n-T_nx_{n+1},j_n\rangle|\\ &\le\|T_ny_n-T_nx_{n+1}\|\cdot\|j_n\|\to 0\quad\text{as} n\to\infty.\end{aligned}$$

On the other hand, since $\sum_{n=1}^{\infty}\beta_n=\infty$ and $\beta_n\to 0$ as $n\to\infty$, we can choose n_1 ($\ge n_0$) so that $\beta_n>0$, $1-2\beta_n(1-k_\epsilon)>0$, and $2k_\epsilon-\beta_n>0$ for all $n\ge n_1$. Then, the above inequality (2.1) can be written as follows:

$$\begin{aligned}&\|x_{n+1}-q\|^2\\ \le&\frac{\alpha_n^2\|x_n-q\|^2}{1-2\beta_n(1-k_\epsilon)}+\frac{2\beta_nd_n}{1-2\beta_n(1-k_\epsilon)}+\frac{2\gamma_nM^2}{1-2\beta_n(1-k_\epsilon)}\\ \le&\frac{(1-\beta_n)^2\|x_n-q\|^2}{1-2\beta_n(1-k_\epsilon)}+\frac{2\beta_nd_n}{1-2\beta_n(1-k_\epsilon)}+\frac{2\gamma_nM^2}{1-2\beta_n(1-k_\epsilon)}\end{aligned}\tag{2.2}$$

Since $\frac{2k_\epsilon - \beta_n}{1-2\beta_n(1-k_\epsilon)} \to 2k_\epsilon$ as $n \to \infty$ and $k_\epsilon \in (0,1)$, there exists a n_2 ($\geq n_1$) such that

$$\left|\frac{2k_\epsilon - \beta_n}{1-2\beta_n(1-k_\epsilon)} - 2k_\epsilon\right| \leq k_\epsilon$$

for all $n \geq n_2$. This implies that $k_\epsilon \leq \frac{2k_\epsilon - \beta_n}{1-2\beta_n(1-k_\epsilon)}$, that is,

$$\frac{(1-\beta_n)^2}{1-2\beta_n(1-k_\epsilon)} \leq (1-k_\epsilon\beta_n)$$

for all $n \geq n_2$. The inequality (2.2) can be expressed as follows.

$$\|x_{n+1} - q\|^2 \leq (1-k_\epsilon\beta_n)\|x_n - q\|^2 + \frac{2\beta_n d_n}{1-2\beta_n(1-k_\epsilon)} + \frac{2\gamma_n M^2}{1-2\beta_n(1-k_\epsilon)},$$

for all $n \geq n_2$. Then it follows from Lemma 1.3 that the sequence $\{x_n\}$ strongly converges to the unique fixed point q of T. Finally, we prove that $F(T) = \{q\}$, a singleton set. If $p \in F(T)$, by using the property (H), we obtain

$$\begin{aligned}\|p - q\|^2 &= \langle p - q, j\rangle \\ &= \limsup_{n\to\infty}\langle T_n p - q, j\rangle \\ &\leq \frac{1}{t}\|p - q\|^2,\end{aligned}$$

for $j \in J(p-q)$, Since $t > 1$, we have $q = p$. □

Remark. In view of the examples 1.1 and 1.2, the above theorem is a new approach of the strong convergence problems of iterative sequences to the unique fixed point of discontinuous non-Lipschitzian self-mappings which are not strictly hemicontractive (hence, not strictly pseudocontractive).

Taking $\beta'_n = \gamma'_n = 0$ for all $n \geq 1$ in (1.6), as a direct consequence of Theorem 2.1, we have the following

Corollary 2.1. *Let K be a nonempty closed convex subset of a Banach space X. Suppose a family $\Im = \{T_n : n \in \mathbb{N}\}$ of self-mappings $T_n : K \to K$, $n \in \mathbb{N}$ satisfies the property* (H) *type. Suppose $F(T) \neq \emptyset$ and put*

$$c_n = \max\{0, \sup_{x,y\in K}(\|T_n x - T_n y\| - \|x - y\|)\},$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then the modified Mann iterative sequence $\{x_n\}_{n=1}^{\infty}$ with errors generated by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad x_1 \in K$$

with $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1]$ satisfying

$$\sum_{n=1}^{\infty}\beta_n = \infty, \quad \sum_{n=1}^{\infty}\gamma_n < \infty, \quad \text{and} \quad \lim_{n\to\infty} b_n = 0,$$

strongly converges $q \in F(T)$ and $F(T)$ is a singleton set.

As a direct consequence of Theorem 2.1, we obtain the following

Theorem 2.2. *Let K be a nonempty bounded closed convex subset of a Banach space X. Suppose a family $\Im = \{T_n : n \in \mathbb{N}\}$ of Lipschitzian self-mappings $T_n : K \to K$, $n \in \mathbb{N}$ satisfies the property* (H) *type. Suppose $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty}(L_n - 1) < \infty$, where $L_n(\geq 1)$ is the Lipschitz constant of T_n. Then the modified Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ with errors generated by (1.6) converges strongly to the unique fixed point of T in K, where*

(i)
$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \beta'_n = \lim_{n\to\infty} \gamma'_n = 0;$$

(ii)
$$\sum_{n=1}^{\infty} \beta_n = \infty \quad and \quad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Proof. Note that

$$\begin{aligned} c_n &= \max\{0, \sup_{x,y\in K} (\|T_n x - T_n y\| - \|x - y\|)\} \\ &\leq (L_n - 1)\delta(K), \end{aligned}$$

where $\delta(K)$ denotes the diameter of K. Note that all assumptions of Theorem 2.1 are fulfilled, □

Taking $\beta'_n = \gamma'_n = 0$ for all $n \geq 1$ in (1.6), as a direct consequence of Theorem 2.2, we have the following

Corollary 2.2. *Let K be a nonempty bounded closed convex subset of a Banach space X. Suppose a family $\Im = \{T_n : n \in \mathbb{N}\}$ of Lipschitzian self-mappings $T_n : K \to K$, $n \in \mathbb{N}$ satisfies the property* (H) *type. Suppose $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty}(L_n - 1) < \infty$, where $L_n(\geq 1)$ is the Lipschitz constant of T_n. Then the modified Mann iterative sequence $\{x_n\}_{n=1}^{\infty}$ with errors generated by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad x_1 \in K$$

with $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1]$ satisfying

$$\sum_{n=1}^{\infty} \beta_n = \infty, \sum_{n=1}^{\infty} \gamma_n < \infty, \quad and \quad \lim_{n\to\infty} b_n = 0,$$

strongly converges $q \in F(T)$ and $F(T)$ is a singleton set.

Remark. Note that if each $T_n : K \to K$ is L_n-Lipschitzian with $\limsup_{n\to\infty} L_n < 1$, then $\Im = \{T_n : n \in \mathbb{N}\}$ is of (H) type.

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Division of Mathematical Sciences, Pukyong National University, Pusan 608-737, Korea
E-mail address: `taehwa@dolphin.pknu.ac.kr`